指数映射：

固定 $p \in D$，对 $\forall\, v \in T_pD$，存在唯一测地线

$\gamma(v, s)$ 使得

- s 为 $\gamma(v,s)$ 弧长参数，在 $0 \le s < \varepsilon_0$ 有定义

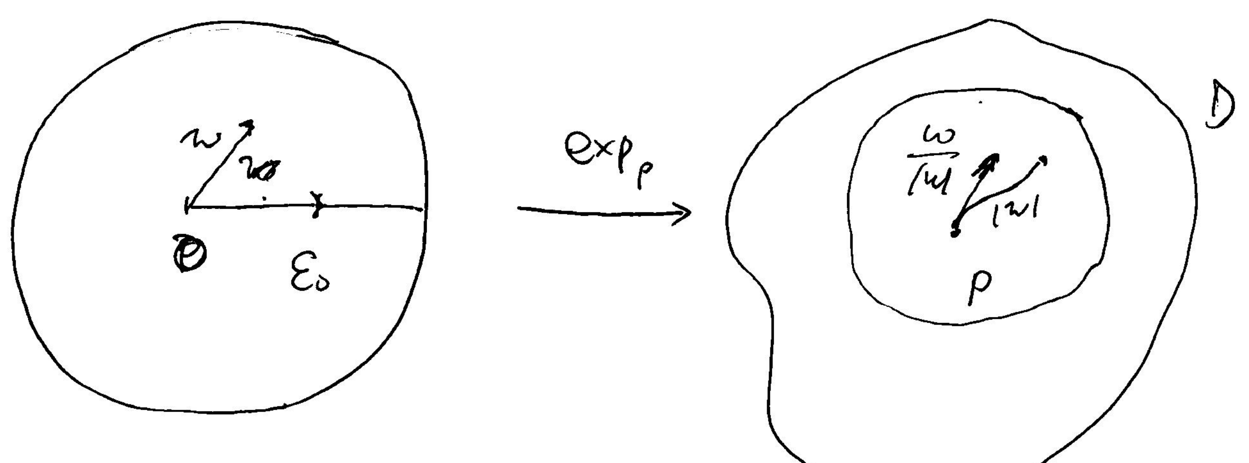
（可以取 ε_0 不依赖 v 的选取）

- $\gamma(v, 0) = p$

- $\dot\gamma(v, 0) = v$

定义：

$$\begin{array}{ccc} T_pD & & \\ \cup & & \\ B_0(\varepsilon_0) & \xrightarrow{\exp_p} & D \\ \Psi & & \Psi \\ w & \longmapsto & \gamma\left(\frac{w}{|w|}, |w|\right) \triangleq \exp_p(w) \end{array}$$



注意到：选定 $e_1(p), e_2(p)$ 在 p 处正交标架，则

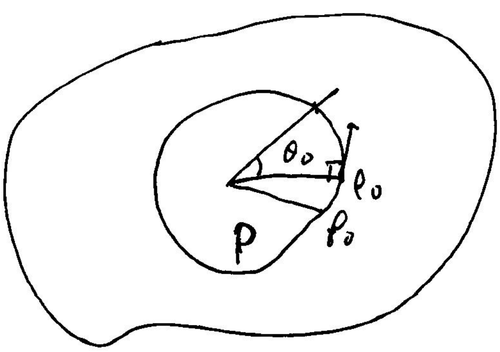
$w = x^1 e_1 + x^2 e_2$，$\{x^1, x^2\}$ 是 $B_0(\varepsilon_0)$ 上的直角坐标系。

利用 $\exp_p$, $\{x^1, x^2\}$ 成为 D 在 P 点邻域的坐标系. (详见后节)

这坐标系称为 P 点 ~~处~~ 为原点的 法坐标系.

令
$$\begin{cases} x^1 = \rho\cos\theta \\ x^2 = \rho\sin\theta. \end{cases}$$

则 $\{\rho, \theta\}$ 称为点 P 为原点的 测地极坐标系 (这坐标系须把 P 点除去)



$\theta = \theta_0$ 是测地线, 以 $v = \cos\theta_0 e_1 + \sin\theta_0 e_2$ 为它的切方向

$\rho = \rho_0$ 称为测地圆. 是由 P 点处所有出发的测地线上离 P 点距离 ρ_0 的点的轨迹.

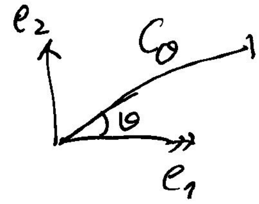
命题: $\{x^1, x^2\}$ 是曲面在 P 处的参数. (即是坐标系)

证明: 设 $\{u^1, u^2\}$ 为 D 在 P 处附近的正交参数. 且

$$e_1 = \frac{\partial}{\partial u^1}(P), \qquad e_2 = \frac{\partial}{\partial u^2}$$

我们要证明

$$\det\left(\frac{\partial u^\alpha}{\partial x^\beta}\right)(p)\neq 0$$



令 C_θ = 与 e_1 成 θ 角的测地线，则测地线方程有

$\| \; (\bar u^1(\rho), \bar u^2(\rho))$，$\rho$ = 弧长参数

$$\boxed{\frac{d^2u^\alpha}{d\rho^2}+\Gamma^\alpha_{\beta\gamma}\frac{du^\beta}{d\rho}\frac{du^\gamma}{d\rho}=0,\qquad \alpha=1,2} \quad (*)$$

「证明： $I=g_{\alpha\beta}\,du^\alpha du^\beta$，则在自然标架下的 Levi-Civita 联络表达式

可是： $\nabla\left(\frac{\partial}{\partial u^\alpha}\right)=\Gamma^\gamma_{\alpha\beta}\,du^\beta\frac{\partial}{\partial u^\gamma}$，$\alpha=1,2$；$\dot C_\theta(\rho)=\dot u^\alpha\frac{\partial}{\partial u^\alpha}$

所以测地线方程 $\nabla_{\dot C_\theta(\rho)}\dot C_\theta(\rho)\equiv 0$ 写为

$$\ddot u^\alpha\frac{\partial}{\partial u^\alpha}+\dot u^\alpha\,\Gamma^\gamma_{\alpha\beta}\,\dot u^\beta\frac{\partial}{\partial u^\gamma}=0$$

$$\Longleftrightarrow\quad \ddot u^\alpha+\Gamma^\alpha_{\beta\gamma}\,\dot u^\beta\dot u^\gamma=0,\quad \alpha=1,2$$ 」

注意到： C_θ 在 p 处的切向量

$$\left(\frac{du^1}{d\rho}e_1+\frac{du^2}{d\rho}e_2\right)\Big|_{\rho=0}=\cos\theta\, e_1+\sin\theta\, e_2$$

所以

$$\frac{du^1}{d\rho}\Big|_{\rho=0}=\cos\theta \;;\quad \frac{du^2}{d\rho}\Big|_{\rho=0}=\sin\theta$$

将 $u^\alpha(\rho)$ 在 $\rho=0$ 处展开：

$$u^\alpha(\rho)=u^\alpha(0)+\frac{du^\alpha}{d\rho}\Big|_{\rho=0}\rho+\frac{1}{2}\left(\frac{d^2u^\alpha}{d\rho^2}\Big|_{\rho=0}\right)\rho^2+\cdots$$

$\|(*)$

$$=0+x^\alpha+\frac{1}{2}\left(-\Gamma^\alpha_{\beta\gamma}\frac{du^\beta}{d\rho}\frac{du^\gamma}{d\rho}\Big|_{\rho=0}\right)\rho^2+\cdots$$

$$=x^\alpha-\frac{1}{2}\Gamma^\alpha_{\beta\gamma}x^\beta x^\gamma+\cdots$$

故 $\left(\frac{\partial u^\alpha}{\partial x^\beta}\right)(\rho=0)=(\delta^\alpha_\beta)$ ⇒ 命题成立. #

$\forall\ \theta_0\in[0,2\pi]$

$$\begin{cases}x^1=\rho\cos\theta_0\\ x^2=\rho\sin\theta_0\end{cases}$$ 是测地线

由上述知 $\{x^1,x^2\}$ 为坐标系，故有

$$\frac{d^2x^\alpha}{d\rho^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\rho}\frac{dx^\gamma}{d\rho}\equiv 0$$ 其中 $\Gamma^\alpha_{\beta\gamma}$ 以 x^1,x^2 为参量.

$$\Rightarrow \quad \Gamma^{\alpha}_{\beta\gamma}(\rho\cos\theta, \rho\sin\theta)\frac{dx^{\beta}}{d\rho}\frac{dx^{\gamma}}{d\rho} = 0, \quad \alpha = 1, 2$$

沿 $\theta = \theta_0$ 成立.

$$\xRightarrow{\rho\to 0} \quad \Gamma^{\alpha}_{\beta\gamma}(p)\left.\frac{dx^{\beta}}{d\rho}\right|_{\rho=0}\left.\frac{dx^{\gamma}}{d\rho}\right|_{\rho=0} \equiv 0, \quad \forall \theta_0$$

简单可验

$$\Rightarrow \quad \Gamma^{\alpha}_{\beta\gamma}(p) = 0, \quad \forall \alpha, \beta, \gamma$$

定理: $\{x^1, x^2\}$ 为法坐标系. 则在此

$$I = g_{\alpha\beta}\,dx^{\alpha}dx^{\beta}, \quad 满足$$

$$(g_{\alpha\beta})(p) = (\delta_{\alpha\beta}), \qquad \frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}(p) = 0, \quad \forall \alpha, \beta, \gamma$$

证明: 显然.

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由于 $\dfrac{\partial(x^1, x^2)}{\partial(\rho, \theta)} = \rho$. 故

$\{\rho, \theta\}$ 为 p 点附近 (除 p 点外) 的坐标系.

定理: (1) $I = d\rho^2 + G(\rho, \theta)\,d\theta^2$

(2) $\lim\limits_{\rho\to 0}\sqrt{G} = 0$

(3) $\lim\limits_{\rho\to 0}(\sqrt{G})_{\rho} = 1$.

证明: 设

$$I = E\,d\rho^2 + 2F\,d\rho\,d\theta + G\,d\theta^2$$

由于 ρ = 弧长参数, 则 $E \equiv 1$

由于 $\theta = \theta_0$ 为测地线.

$$\frac{d^2u^\alpha}{d\rho^2} + \Gamma^\alpha_{\beta\gamma}\frac{du^\beta}{d\rho}\frac{du^\gamma}{d\rho} = 0 \qquad \begin{cases} u^1 = \rho \\ u^2 = \theta = \theta_0 \end{cases}$$

$$\Rightarrow \Gamma^2_{11} = 0$$

$$\Gamma^2_{11} = \frac{1}{EG-F^2}\Big(-\frac{F}{2}\underbrace{\frac{\partial E}{\partial u^1}}_{=0} \overbrace{-\frac{E}{2}}^{-\frac{E}{2}}\underbrace{\frac{\partial E}{\partial u^2}}_{=0} + E\underbrace{\frac{\partial F}{\partial u^1}}_{=\frac{\partial F}{\partial \rho}}\Big)$$

$$\Rightarrow \frac{\partial F}{\partial \rho} = 0, \quad 即\ F(\rho,\theta) = \lim_{\rho\to 0} F(\rho,\theta)$$

但 $\frac{\partial}{\partial\theta} = -\rho\sin\theta\frac{\partial}{\partial x^1} + \rho\cos\theta\frac{\partial}{\partial x^2}$; $\frac{\partial}{\partial\rho} = \cos\theta\frac{\partial}{\partial x^1} + \sin\theta\frac{\partial}{\partial x^2}$

$$\lim_{\rho\to 0} F(\rho,\theta) = \lim_{\rho\to 0}\left\langle \frac{\partial}{\partial\rho}, \frac{\partial}{\partial\theta}\right\rangle = \left\langle \cos\theta\frac{\partial}{\partial x^1} + \sin\theta\frac{\partial}{\partial x^2}, \underbrace{\lim_{\rho\to 0}\frac{\partial}{\partial\theta}}_{=0}\right\rangle$$

$$= 0$$

$$\Rightarrow F \equiv 0.$$

另一方面: 令 $g_{\alpha\beta} = g_{\alpha\beta}(x^1, x^2)$, 则有

$$\sqrt{G} = \sqrt{EG-F^2} = \frac{\partial(x^1,x^2)}{\partial(\rho,\theta)}\sqrt{\det(g_{\alpha\beta})} = \rho\sqrt{\det(g_{\alpha\beta})}$$

$$\Rightarrow \lim_{\rho \to 0} \sqrt{G} = 0$$

最后，因为

$$\left(\sqrt{G}\right)_\rho = \sqrt{\det(g_{\alpha\beta})} + \rho \frac{\partial}{\partial \rho} \sqrt{\det(g_{\alpha\beta})}$$

$$= \sqrt{\det(g_{\alpha\beta})} + \rho \frac{1}{2\sqrt{\det(g_{\alpha\beta})}} \left(\cos\theta \frac{\partial}{\partial \chi^1} (\det(g_{\alpha\beta})) + \sin\theta \frac{\partial}{\partial \chi^2} (\det(g_{\alpha\beta})) \right)$$

$$\Rightarrow \lim_{\rho \to 0} \left(\sqrt{G}\right)_\rho = \lim_{\rho \to 0} \sqrt{\det(g_{\alpha\beta})} + \rho \, \boxed{\quad}$$

$$= 1 + 0$$

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